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ON CONVECTIVE INSTABILITY OF AN INCLINED FLUID LAYER EQUILIBRIUM RELATIVE TO SPATIAL PERTURBATIONS

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The convective instability of an inclined fluid layer equilibrium relative to normal spatial perturbations is considered. It is found that spatial perturbations in an inclined layer are the most dangerous, which is supported by experimental results [1].

A similar problem of stability involving plane perturbations was earlier considered in [2,3].

1. Amplitude equations. Let us consider a plane layer of fluid of thickness $2h$ inclined at angle α to the vertical. The disposition of coordinate axes is shown in Fig. 1, where the y-axis is horizontal. The layer is heated from below in such a way that the mechanical equilibrium of the fluid is ensured, and the temperature gradient is constant and vertical

$$\nabla T_0 = -A\gamma \tag{1.1}$$

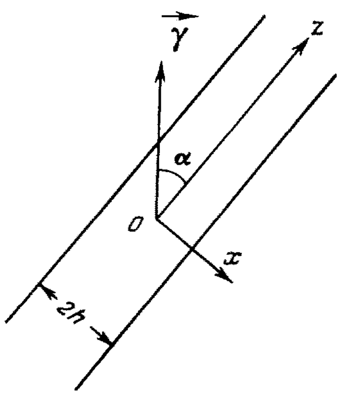

Fig. 1

The dimensionless equations for small neutral equilibrium perturbations are conventionally obtained from convection equations in the Boussinesq approximation. These equations are of the form

$$\begin{aligned} &\Delta \mathbf{v} + RT\gamma = \nabla p, \\ &\Delta T = -(\mathbf{v}\,\gamma), \quad \operatorname{div} \mathbf{v} = 0 \\ &R = g\beta A h^4 / (\nu\chi) \end{aligned} \tag{1.2}$$

where measurement units are the same as in [2] and R is the Rayleigh number.

We consider normal spatial perturbations of the form

$$(v_x, v_y, v_z, T, p) \sim \exp i\,(k_y y + k_z z) \tag{1.3}$$

The system of amplitude equations can be derived from (1.2). Eliminating from that system amplitudes v_y and v_z, and p, for the amplitude of the transverse component of velocity u, and temperature θ we obtain the system of eighth order equations

$$\begin{aligned} &\Delta^2 u - ikaR\cos\alpha\theta' + k^2 R \sin\alpha\theta = 0 \\ &\Delta^2\theta + (a^2 - 1)\,R\cos^2\alpha\theta - \sin\alpha\Delta u + \frac{ia}{k}\cos\alpha\Delta\, u' = 0 \end{aligned} \tag{1.4}$$

$$\Delta = d^2 / dx^2 - k^2, \quad k = \sqrt{k_y^2 + k_z^2}, \quad a = k_z / k$$

where the prime indicates differentiation with respect to the transverse coordinate x, k is the wave vector module, and a is the parameter of spatial perturbations. We assume that the layer boundaries are solid and that along these the temperature distribution is linear with respect to z, which ensures the fulfillment of the equilibrium condition (1.1). For the amplitude of perturbations of u and θ we now have the homogeneous boundary conditions

$$u = u' = 0, \quad \theta = \theta'' = 0, \quad x = \pm 1 \tag{1.5}$$

The boundary value problem (1.4), (1.5) determines the spectrum of critical values of the Rayleigh number R. The angle of inclination α and the characteristic of spatial perturbations of k and a are taken as parameters. The limit case of plane perturbations considered in [2] corresponds to $k_y = 0$, i.e. $k = k_z$ and $a = 1$.

2. Long-wave perturbations. In the limit case of long-wave perturbations $(k \ll 1)$ the eigenfunctions and eigenvalues of problem (1.4),(1.5) may be sought in the form of expansions in powers of the small parameter (ik)

$$u = iku_0 + (ik)^2u_1 + (ik)^3u_2 + \ldots \tag{2.1}$$
$$\theta = \theta_0 + ik\theta_1 + (ik)^2\theta_2 + \ldots, \quad R = R_0 + k^2R_2 + \ldots$$

Substituting these expansions into (1.4) we obtain the system of successive approximations

$$u_0^{IV} - aR_0 \cos \alpha\theta_0' = 0 \tag{2.2}$$
$$\theta_0^{IV} + (a^2 - 1) R_0 \cos^2 \alpha\theta_0 - a \cos \alpha u_0''' = 0$$

$$u_1^{IV} - aR_0 \cos \alpha\theta_1' = R_0 \sin \alpha\theta_0 \tag{2.3}$$
$$\theta_1^{IV} + (a^2 - 1) R_0 \cos^2\alpha\theta_1 - a \cos \alpha u_1''' = \sin \alpha\, u_0''$$

$$u_2^{IV} - aR_0 \cos \alpha\theta_2' = -2u_0'' - aR_2 \cos \alpha\theta_0' + R_0 \sin \alpha\theta_1 \tag{2.4}$$
$$\theta_2^{IV} + (a^2 - 1) R_0 \cos^2\alpha\theta_2 - a \cos \alpha u_2''' = -2\theta_0'' +$$
$$(a^2 - 1) R_2 \cos^2\alpha\theta_0 + \sin \alpha u_1'' + a \cos \alpha u_0'$$

Boundary conditions for amplitudes of various orders coincide with (1.5).

System (2.2) with related boundary conditions determines the critical Rayleigh number and the form of perturbations with $k = 0$ and arbitrary a. In the zero approximation the amplitude problem has two solutions which differ by their eigenfunction parity. Solutions normalized in a specific manner, which are subsequently called "even", are of the form

$$u_0 = -\frac{a\gamma \cos \gamma}{\cos \alpha}\left[\frac{\sin \gamma x}{\cos \gamma} - \frac{\operatorname{sh} \gamma x}{\operatorname{ch} \gamma} + \gamma^3 \frac{a^2 - 1}{3a^2}(x^3 - 3x)\right] \tag{2.5}$$
$$\theta_0 = \cos \gamma \left[\frac{\cos \gamma x}{\cos \gamma} + \frac{\operatorname{ch} \gamma x}{\operatorname{ch} \gamma} - 2\right], \quad \gamma^4 = R_0 \cos^2 \alpha$$

The eigenvalues of R_0 are determined by the transcendental equation

$$\operatorname{tg}\gamma - \operatorname{th}\gamma = \frac{2}{3}\gamma^3\frac{a^2-1}{a^2} \tag{2.6}$$

The "odd" solutions are determined by formulas

$$u_0 = \frac{a\gamma\cos\gamma}{\cos\alpha}\left(\frac{\cos\gamma x}{\cos\gamma} - 1\right), \quad \theta_0 = \sin\gamma x, \quad \gamma^4 = R_0\cos^2\alpha \tag{2.7}$$

The eigenvalues for this part of the spectrum are independent of parameter a and are determined by the equation

$$\sin\gamma = 0; \quad \gamma = \pi,\ 2\pi,\ \ldots; \quad R_0 = \frac{\pi^4}{\cos^2\alpha},\ \frac{(2\pi)^4}{\cos^2\alpha},\ \ldots \tag{2.8}$$

Thus the critical Rayleigh number for instability with respect to perturbations with $k = 0$ is determined by formula

$$R_0 = \gamma^4 / \cos^2\alpha \tag{2.9}$$

where γ is the lower root of the characteristic equation (2.6) or (2.8). The lower root of (2.6) depends on a. When $a = 0$ we have $\gamma = \pi / 2$, with increasing a the critical number γ monotonically increases, and $\gamma \to 3.927$ when $a \to 1$. The lower root (2.8) is independent of a and is equal to π. The minimal critical number γ is shown in Fig. 2 by solid lines as a function of a. It will be seen that in the region $a < a_* = 0.9767$ the most dangerous perturbations with $k = 0$ are of the even type, while for $a > a_*$ they are of the odd type.

To decide whether among long-wave perturbations the most dangerous are those with $k = 0$ it is necessary to know the sign of the quadratic correction R_2 in expansion (2.1). When $R_2 > 0$ perturbations with $k = 0$ yield the minimum critical Rayleigh number, hence they are the most dangerous; when $R_2 < 0$ point $k = 0$ corresponds to the maximum on curve $R(k)$, and the most dangerous are then the "cellular" perturbations with $k \neq 0$.

The quadratic correction R_2 is determined by the condition of solvability of the nonhomogeneous system (2.4) of second approximation. To establish the condition of solvability it is necessary to know the solution of the systems of zero and first order of (2.2) and (2.3), as well as the solution of the homogeneous problem conjugate of (2.2). The condition $R_2(\alpha, a) = 0$ makes possible the determination of the critical angle of inclination $\alpha_c(a)$ for which the form of instability changes: a transition from perturbations with $k = 0$ to cellular perturbations takes place.

The formulas for corrections R_2 and for critical angles α_c in both the even and odd cases are very cumbersome and are not presented here. We only show the curve that defines the dependence of the critical angle α_c on the parameter of spatial perturbations a (Fig. 2). In the limit case $a = 1$ ($k_y = 0$; for plane perturbations see [2]) $\alpha_c = 20°46'$. In the opposite limit case of $a = 0$, which corresponds to $k_z = 0$ and $k_y \neq 0$ ("helical" perturbations of the kind of shafts stretched along the z-axis), $\alpha_c = 72°53'$. The curve in Fig. 2 makes possible the determination of the critical angle for any arbitrary value of parameter a. At the point $a = a_*$, where the change of "evenness" of the stability level occurs, α_c vanishes.

3. Numerical solution for arbitrary k. Formula (2.9) yields the minimal critical Rayleigh number (with respect to k) that determines the stability limit in the region of angles $\alpha < \alpha_c$. When $\alpha > \alpha_c$ the absolute minimum on neutral curves $R\ (k;\ \alpha,\ a)$ is reached at finite wave numbers. To determine in that case the limits of stability the boundary value problem (1.4), (1.5) was solved numerically. The amplitude equations (1.4) were reduced to 16 first order real equations, and the system of these equations was integrated by the Runge — Kutta method.

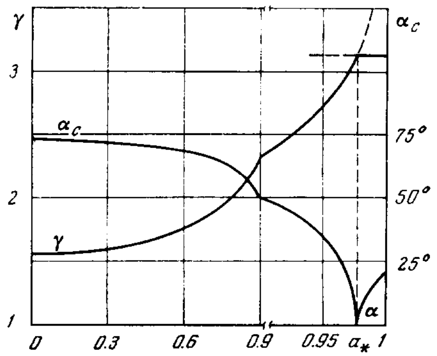

Fig. 2

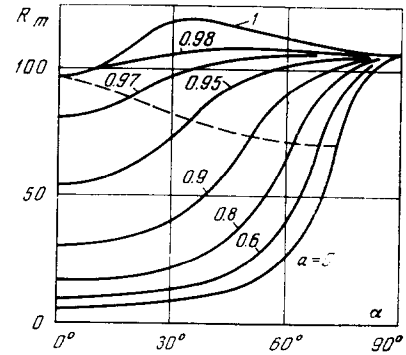

Fig. 3

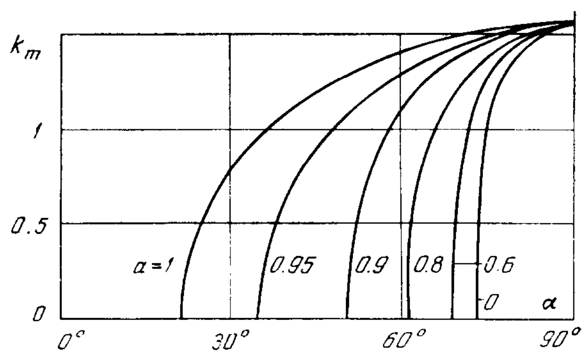

Fig. 4

The minimization of eigenvalues of R with respect to k obtained in this way yielded the minimal critical number $R_m(\alpha,\ a)$ and the wave number $k_m\ (\alpha,\ a)$ of the most dangerous perturbations.

The results appear in Figs. 3 and 4. The set of curves in Fig. 3 represents the dependence of the minimal critical Rayleigh number R_m on the angle of inclination to the vertical. Transition to cellular structures is indicated by the dash line, below which for $a < a_*$ perturbations with $k_m = 0$ are responsible for instability. Sections of curves above the dash line relate to cellular perturbations with $k_m \neq 0$. When $a > a_*$ curves $R_m\ (\alpha)$ in the range $0 < \alpha < \alpha_c\ (a)$ are the same for various a . It is evident that throughout the range of angles spatial perturbations are more dangerous than plane ones, and that the absolute minimum of the critical Rayleigh number is produced by helical perturbations $(a = 0)$. The significant decrease of stability for a relatively small deviation of perturbations from the plane structure is noteworthy. When $\alpha \rightarrow 90^\circ$ "degeneration" takes place: in a horizontal layer plane and spatial preturbations correspond to the same critical Rayleigh number $R_m =$ 106.7 (when the total thickness of the layer is taken as the characteristic length $R_m = 1708$).

The dependence of k_m on α is shown in Fig. 4 in the form of curves for several spatial perturbation parameters $a < a_*$, including the curve for plane perturbations $(a = 1)$.

Experiments had shown [1] that the crisis of equilibrium in an inclined layer is almost throughout the range of angles related to spatial perturbations. A qualitative comparison of calculated and experimental results is, unfortunately, difficult, since the former relate to perfectly conducting boundary surfaces, while the experimental results relate to a layer of kerosene between Plexiglas plates (thermal conductivity ratio $\varkappa \approx 0.7$).

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